Modeling Debt
Scheduled Debt Amortization

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In cases such as loan guarantees the Guarantor is responsible for the Borrower's outstanding debt balance (unamortized loan principal plus accrued interest) at the end of the guarantee term if the Borrower defaults on that debt sometime over the time interval $[0, \text{Guarantee term}]$. To facilitate loan guarantee modeling we need a model for debt balance in discrete-time. To that end we will work through the following hypothetical problem...

Our Hypothetical Problem

We are given the following model parameters...

Table 1: Model Parameters

Symbol	Description	Value
D_0	Debt principal balance at time zero (\$)	100,000
B	End-of-term balloon payment (\$)	25,000
R	Annual contractual interest rate (%)	6.00
N	Number of annual time periods (#)	12
T	Total number of time periods (#)	60

Our task is to answer the following questions...

Question 1: Reconcile the change in debt in year 2.

Question 2: The borrower defaults on day 452 of the debt term. What is the Guarnator's obligation under the terms of the guarantee?

Debt Equations

We will define the variable μ to be the periodic discrete-time interest rate, which is the interest rate applicable to each discrete time period. Using the data in Table 1 above the equation for the periodic interest rate is...

$$\mu = \frac{R}{N} = \frac{0.0600}{12} = 0.0050 \tag{1}$$

We will define the variable θ to be the periodic discount factor, which is the discount factor applicable to each discrete time period. Using Equation (1) above the equation for the periodic discount factor is...

$$\theta = \frac{1}{1+\mu} = \frac{1}{1+0.0050} = 0.9950 \tag{2}$$

Using the Equation (2) above the solution to the following polylogarithm (i.e. geometric series) is... [1]

$$\sum_{m=1}^{n} \theta^m = \frac{\theta^m - \theta^{n+1}}{1-\theta} = \frac{\theta\,(1-\theta^n)}{1-\theta} \tag{3}$$

We will define the variable D_0 to be debt balance at time zero and the variable P to be the periodic debt service payment amount. Using Equation (2) above and the data in Table 1 above the equation for debt balance at time zero is...

$$D_0 = \sum_{m=1}^{T} P\,\theta^m + B\,\theta^T \tag{4}$$

Using Appendix Equation (24) below the solution to Equation (4) above is...

$$D_0 = P\,\frac{\theta\,(1-\theta^T)}{1-\theta} + B\,\theta^T \tag{5}$$

Given that we know debt balance at time zero, balloon payment at time T, and the value of the discount factor θ, we can solve Equation (5) above for the variable P. The equation for the periodic debt service payment is...

$$P = \frac{(1-\theta)\,(D_0 - B\,\theta^T)}{\theta\,(1-\theta^T)} \tag{6}$$

Using Equations (5) and (6) above the equation for debt balance at the end of time period $t < T$ is...

$$D_t = \sum_{m=1}^{T-t} P\,\theta^m + B\,\theta^{T-t} \tag{7}$$

Using Appendix Equation (26) below the solution to Equation (7) above is...

$$D_t = P\,\frac{\theta\,(1-\theta^{T-t})}{1-\theta} + B\,\theta^{T-t} \tag{8}$$

Using the data in Table 1 above the equation for debt APR (annual percentage rate) is...

$$\text{Debt APR} = \left(1.00 + \frac{R}{N}\right)^N - 1.00 = \left(1.00 + \frac{0.0600}{12}\right)^{12} - 1.00 = 6.17\% \tag{9}$$

Scheduled Debt Amortization

We will define the variable $I_{a,b}$ to be cumulative interest earned over the period interval $[a,b]$. Using Equations (1) and (8) above the equation for cumulative interest is...

$$I_{a,b} = \sum_{t=a-1}^{b-1} \mu\,D_t \tag{10}$$

Using Appendix Equation (31) below the solution to Equation (10) above is...

$$I_{a,b} = \frac{\mu\,P\,\theta\,(b-a+1)}{1-\theta} + \mu\,\lambda^{a-2}\left(B\,\theta^T - \frac{P\,\theta^{T+1}}{1-\theta}\right)\frac{\lambda\,(1-\lambda^{b-a+1})}{1-\lambda} \quad \text{...where...} \quad \lambda = \frac{1}{\theta} \tag{11}$$

We will define the variable $J_{a,b}$ to be cumulative debt service payments received over the period interval $[a,b]$.

$$J_{a,b} = \sum_{t=a}^{b} P = P\,(b-a+1) \tag{12}$$

Using Equations (11) and (12) above the equation for principal amortization over the period interval $[a,b]$ is...

$$\text{Principal amortization} = D_b - D_{a-1} = J_{a,b} - I_{a,b} \tag{13}$$

The Answers To Our Hypothetical Problem

Question 1: Reconcile the change in debt in year 2.

Using Equations (2) and (6) above and the data in Table 1 above the periodic debt payment for our problem is...

$$P = \frac{(1-0.9950)\times(100,000 - 25,000\times 0.9950^{60})}{0.9950\times(1-0.9950^{60})} = 1,574.96 \tag{14}$$

Using Equations (2), (8) and (14) above and the data in Table 1 above scheduled debt balance at the end of period 12 is...

$$D_{12} = 1,574.96 \times \frac{0.9950\times(1-0.9950^{60-12})}{1-0.9950} + 25,000\times 0.9950^{60-12} = 86,739.76 \tag{15}$$

Using Equations (2), (8) and (14) above and the data in Table 1 above scheduled debt balance at the end of period 24 is...

$$D_{24} = 1,574.96 \times \frac{0.9950 \times (1 - 0.9950^{60-24})}{1 - 0.9950} + 25,000 \times 0.9950^{60-24} = 72,661.66 \tag{16}$$

Using Equation (2) and (11) the equation for the variable λ is...

$$\lambda = \frac{1}{0.9950} = 1.0050 \tag{17}$$

Using Equations (1), (2), (14), (17) above and the data in Table 1 above interest accrued in year 2 is...

$$I_{13,24} = \frac{0.0050 \times 1,574.96 \times 0.9950 \times (24 - 13 + 1)}{1 - 0.9950} + 0.0050 \times 1.0050^{13-2} \times \Bigg(25,000 \times 0.9950^{60} - \frac{1,574.96 \times 1,574.96^{60+1}}{1 - 1,574.96}\Bigg) \times \frac{1.0050 \times (1 - 1.0050^{24-13+1})}{1 - 1.0050} = 4,821.42 \tag{18}$$

Using Equations (12) and (14)

$$J_{13,24} = 1,574.96 \times (24 - 13 + 1) = 18,899.52 \tag{19}$$

Using the equations above the answer to the question is...

Description	Value	Reference
Beginning debt balance	86,739.76	Equation(15)
Accrued interest	+ 4,821.42	Equation (14)
Debt service payments	- 18,899.52	Equation (15)
Ending debt balance	72,661.66	Equation (16)

Question 2: The borrower defaults on day 452 of the debt term. What is the Guarnator's obligation under the terms of the guarantee?

Using the data in Table 1 above the borrower defaulted at the end of period...

$$\text{Default period} = \text{INT}\Bigg[\frac{\text{default day}}{365} \times N\Bigg] = \text{INT}\Bigg[\frac{452}{365} \times 12\Bigg] = 14 \tag{20}$$

Using Equations (2), (8), (14) and (20) above and the data in Table 1 above scheduled debt balance at the beginning of the default period is...

$$D_{37} = 1,574.96 \times \frac{0.9950 \times (1 - 0.9950^{60-14})}{1 - 0.9950} + 25,000 \times 0.9950^{60-14} = 84,392.18 \tag{21}$$

Using Equations (1) and (21) above the answer to the question is...

$$\text{Guarantor obligation} = \text{Debt principal} + \text{Accrued interest} = 84,392.18 \times \Bigg(1 + 0.0050\Bigg)^{60-14} = 106,229.80 \tag{22}$$

References

[1] Gary Schurman, *Polylogarithms Of Order Zero,* May, 2019.

Appendix

A. The solution to Equation (4) above is...

$$D_0 = \sum_{m=1}^{T} P\,\theta^m + B\,\theta^T = P \sum_{m=1}^{T} \theta^m + B\,\theta^T \tag{23}$$

Using Equation (3) above we can rewrite Equation (23) above as...

$$D_0 = P\,\frac{\theta\,(1-\theta^T)}{1-\theta} + B\,\theta^T \tag{24}$$

B. The solution to Equation (4) above is...

$$D_t = \sum_{m=1}^{T-t} P\,\theta^m + B\,\theta^{T-t} = P\sum_{m=1}^{T-t}\theta^m + B\,\theta^{T-t} \tag{25}$$

Using Equation (3) above we can rewrite Equation (25) above as...

$$D_t = P\,\frac{\theta\,(1-\theta^{T-t})}{1-\theta} + B\,\theta^{T-t} \tag{26}$$

C. The solution to Equation (10) above is...

$$\begin{aligned} I_{a,b} &= \mu\sum_{t=a-1}^{b-1}\left[P\,\frac{\theta\,(1-\theta^{T-t})}{1-\theta} + B\,\theta^{T-t}\right] \\ &= \mu\left[\frac{P\,\theta}{1-\theta}\sum_{t=a-1}^{b-1}(1-\theta^{T-t}) + B\sum_{t=a-1}^{b-1}\theta^{T-t}\right] \\ &= \mu\left[\frac{P\,\theta}{1-\theta}\sum_{t=a-1}^{b-1}1 - \frac{P\,\theta}{1-\theta}\sum_{t=a-1}^{b-1}\theta^T\theta^{-t} + B\sum_{t=a-1}^{b-1}\theta^T\theta^{-t}\right] \\ &= \mu\left[\frac{P\,\theta\,(b-a+1)}{1-\theta} - \frac{P\,\theta^{T+1}}{1-\theta}\sum_{t=a-1}^{b-1}\theta^{-t} + B\,\theta^T\sum_{t=a-1}^{b-1}\theta^{-t}\right] \\ &= \frac{\mu\,P\,\theta\,(b-a+1)}{1-\theta} + \mu\left(B\,\theta^T - \frac{P\,\theta^{T+1}}{1-\theta}\right)\sum_{t=a-1}^{b-1}\theta^{-t} \end{aligned} \tag{27}$$

We will make the following definition...

$$\lambda = \frac{1}{\theta} \tag{28}$$

Using the definition in Equation (28) above we can rewrite Equation (27) above as...

$$I_{a,b} = \frac{\mu\,P\,\theta\,(b-a+1)}{1-\theta} + \mu\left(B\,\theta^T - \frac{P\,\theta^{T+1}}{1-\theta}\right)\sum_{t=a-1}^{b-1}\lambda^t \tag{29}$$

After adding $2-a$ to the upper and lower summation bounds Equation (29) becomes...

$$\begin{aligned} I_{a,b} &= \frac{\mu\,P\,\theta\,(b-a+1)}{1-\theta} + \mu\left(B\,\theta^T - \frac{P\,\theta^{T+1}}{1-\theta}\right)\sum_{t=1}^{b-a+1}\lambda^t\lambda^{a-2} \\ &= \frac{\mu\,P\,\theta\,(b-a+1)}{1-\theta} + \mu\,\lambda^{a-2}\left(B\,\theta^T - \frac{P\,\theta^{T+1}}{1-\theta}\right)\sum_{t=1}^{b-a+1}\lambda^t \end{aligned} \tag{30}$$

Using Equation (3) above we can rewrite Equation (30) above as...

$$I_{a,b} = \frac{\mu\,P\,\theta\,(b-a+1)}{1-\theta} + \mu\,\lambda^{a-2}\left(B\,\theta^T - \frac{P\,\theta^{T+1}}{1-\theta}\right)\frac{\lambda\,(1-\lambda^{b-a+1})}{1-\lambda} \tag{31}$$